Information-Theoretic Approach to the Study of Control Systems

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***Abstract*—We propose an information-theoretic framework for studying control systems, based on a model of controllers analogous to communication channels. Given the initial state of a system to be controlled, the dynamics of a controller is described as applying a transmission channel, called the actuation channel, to that state in order to redirect it towards another target state. In this process, two different control strategies can be adopted: (i) the controller applies an actuation dynamics independently of the state of the system to be controlled (open-loop control); or (ii) the controller enacts an actuation dynamics based on some information about the state of the controlled system (closed-loop control). In the context of this model, we provide necessary and sufficient entropic conditions for a system to be perfectly controllable and perfectly observable. Also, using the fact that the information gathered by a controller is quantified by mutual information, we prove a limiting result expressing the trade-off between the availability of information in a closed-loop control process and its performance over open-loop control in stabilizing a system. This work completes a first paper on the subject [1] by providing new proofs of the results, and by proposing an information-based optimality criterion for control systems. New applications of this approach pertaining to proportional controllers, and the control of chaotic maps are also presented.**

***Keywords*—Control theory, stochastic systems, entropy, mutual information, communication channel, control channel, stochastic stability, proportional controller, chaotic control.**

I. Introduction

CONVENTIONAL control systems are constructed from two fundamental and usually distinct physical components: *sensors* and *actuators*. On the one hand, sensors are devices whose task, as the word plainly suggests, is to sense, observe or estimate the state of a system intended to be controlled. Actuators, on the other hand, are devices which act directly on the controlled system by augmenting its natural dynamics so as to redirect its evolution toward a desired response, thereby achieving purposeful control. The actual control or actuation dynamics applied can be prescribed either by the outcome of a sensor, or by general characteristics of the controlled system not expressly related to the instant variations of its state. In the control-theoretic jargon, the former control strategy corresponds to what is known as *closed-loop* or *feedback* control, whereas the latter is referred to as *open-loop* control [2], [3].

Intuitively, the functioning of sensors and actuators in a control unit is often described by having recourse to the complementary concepts of uncertainty and information. Sensors can be thought of as gathering information from the system to be controlled in the form of data relative to its state (estimation step); this information is processed according to a determined control strategy (decision step), and then transferred to actuators which feed this information back to the controlled system by modifying its dynamics, with the goal of decreasing the uncertainty about the value of the system's variables (actuation step) [3]. In this spirit, it can be said that an open-loop controller distinguishes itself from a closed-loop controller in that it does not need a continual input of 'selective' information [4] to work: like the throttle of a gas pipe or a blindfolded driver, to take simple examples, it implements a control action independently of the state of the controlled system. In this respect, open-loop control techniques then merely represent a subclass of closed-loop controls restricted by the fact that information made available by estimation is neglected.

In view of this compelling information-based description of control units, it is perhaps surprising to note that few efforts have been made to go beyond the intuitive and qualitative aspects of it to develop a *quantitative* theory of controllers focused explicitly on information. Indeed, although controllers have been described by numerous authors as information gathering and using systems (e.g., [5]-[8]), and despite some results related to this problem (see [9]-[20] and most notably [21]-[24]), there exists at present no general information-theoretic formalism characterizing the exchange of information between a controlled system and a controller, and more importantly, which allows for the assignation of a definite value of information in control processes. To address this deficiency, we proceed in this paper with a detailed study of an attempt for such a formalism elaborated first in [1]. The basis of the results presented here draws upon the work of several of the papers cited above by bringing together some aspects of dynamical systems, information theory, in addition to probabilistic networks to construct control models in the context of which quantities analogous to entropy can be defined.

Central to our approach is the notion of a communication channel, and its extension to the idea of *control channels*. As originally proposed by Shannon [25], a (memoryless) communication channel can be represented mathematically by a probability transition matrix, say $p(y|x)$, relating the two random variables X and Y which are interpreted, respectively, as the input and the output of the channel. In the next two sections of the present work, we adapt this common probabilistic picture of communication engineering to describe the operation of a basic control setup, composed of a sensor linked to an actuator, in terms of two channels: one coupling the initial state of the system to be controlled and the state of the sensor (sensor channel), and another one describing the state evolution of the controlled system as influenced by the sensor-actuator's states (actuation channel).

In Sections IV and V, we use this model in conjunction with the properties of entropy-like quantities to exhibit fundamental

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results pertaining to control systems. As a first of these results, we show that the classical definition of controllability, a concept well-known to the field of control theory, can be rephrased in an information-theoretic fashion. This new definition is used, in turn, to show that a necessary and sufficient condition for a system to be perfectly controllable is that the target state of that system, upon the application of controls, is statistically independent of any other external systems playing the role of noise sources. A similar result is also proven for the complementary concept of observability. Moreover, we prove that the information a feedback controller must gather in order to stabilize the state of an arbitrary system by decreasing its entropy must be bounded below by the difference $\Delta H_{\text{closed}} - \Delta H_{\text{open}}^{\max}$, where ΔH_{closed} is the closed-loop entropy reduction that results from utilizing information in the control process, and $\Delta H_{\text{open}}^{\max}$ is the maximum decrease of entropy attainable when restricted to open-loop control techniques. This last result, as we will see, may be used to define an information-based optimality criterion for control systems.

The idea of reducing the entropy of a system using information gathered from estimating its state is not novel by itself. It has, in fact, been treated abundantly in the physics literature in the context of thermodynamics, particularly in connection with the so-called Maxwell's demon paradox. (See [26] for a description of this paradox and a guide to the original literature.) However, it is an unfortunate fact that familiarity with the Maxwell's demon paradox is not widespread among engineers working in control theory, and therefore discussions of this subject in physics had very limited impact, if none, on the field of control.

To the authors' knowledge, the only published works containing control-oriented findings which exploit on a quantitative level the idea of reducing the entropy of a dynamical system have been reported by Poplavskiĭ [12], [13] and by Weidemann [21]. Although the content of these references is similar in essence to that which is presented here, the conceptual approach adopted by their respective authors fails to address the problem of information and control in its full generality. For instance, most of the results obtained by Poplavskiĭ concerning the information gathered by sensoring devices are based on Brillouin's notion of negentropy, a quantity which proved with time to be very misleading as it gave rise to a number of misconceptions related to the reversibility of information processing. In addition, his study focuses almost entirely on the sensor part of controllers, leaving completely aside the actuation process which, as will be shown, can be also treated in an information-theoretic fashion. In the same vein, the results derived by Weidemann lack a certain generality due to the fact that he only considers a restricted class of linear control systems having measure preserving sensors.

In the present paper, we go beyond these limitations by presenting results which apply equally to linear and nonlinear systems, and can be generalized with the aid of a few modifications to encompass continuous-space systems as well as continuous-time dynamics. To illustrate this scope of applications, we study in Section VI specific examples of control systems. Among these, we consider two variants of proportional controllers, which play a predominant role in the design of present-day controllers, in addition to complete our numerical investigation of noise-perturbed chaotic controllers initiated in [1]. Finally, we propose, by way of conclusion, a general discussion of the relationship of our framework with thermodynamics, optimal control theory, and rate distortion theory.

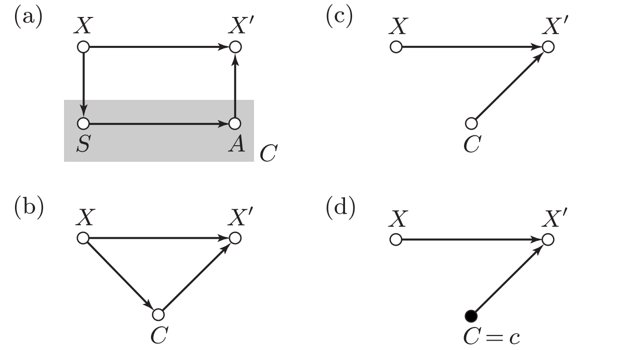

Fig. 1. Directed acyclic graphs representing a basic control process. (a) Full control system with a sensor S and an actuator A. (b) Reduced closed-loop diagram obtained by merging the sensor and the actuator into a single controller device, the controller. (c) Reduced open-loop control diagram. (d) Single actuation channel enacted by the controller's state $C = c$.

II. Channel-like models of control

In this section, we introduce a model that allows investigation of the general control problem in its simplest but nontrivial version. It appears to us that this model, while focusing only on a few components of controllers, nonetheless captures the essence of what a control process is about: that is, a dynamical interplay between a sensor and an actuator aimed at enforcing the dynamics of a system from one initial state to a final target state. In this sense, the proposed model is arguably the best possible compromise between, on the one hand, the desire to address the problem at a level amenable to formalization, and, on the other hand, the need to deduce results from it which are relevant for the study of realistic control systems.

The basic control models that we are interested in are depicted schematically in Figure 1 in the form of directed acyclic graphs, also known as Bayesian networks [27], [28]. The vertices of these graphs correspond to random variables representing the state of a particular (classical) system, whereas the arrows give the probabilistic dependencies among the random variables according to the general decomposition

$$p(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} p(x_i|\pi[X_i]), \tag{1}$$

where $\pi[X_i]$ is the set of random variables which are direct parents of X_i, $i = 1, 2, \ldots, N$, ($\pi[X_1] = \emptyset$). The acyclic condition of the graphs ensures that no vertex is a descendant or an ancestor of itself, in which case we can order the vertices chronologically, i.e., from ancestors to descendants. This defines a causal ordering, and, consequently, a time line directed on the graphs from left to right.

In the control graph of Figure 1a, the random variable X represents the initial state of the system to be controlled, and whose values $x \in \mathcal{X}$ are drawn according to a fixed probability distribution $p_X(x)$. In conformity with our introductory description of controllers, this initial state is controlled to a final state X' with state values $x' \in \mathcal{X}$ by means of a sensor, of state variable S, and an actuator whose state variable A influences the transition from X to X'. For simplicity, all the random variables describing the different systems are taken to be discrete random variables with finite sets of outcomes. The extension to continuous-state systems is discussed in Section IV. Also, to further simplify the analysis of this model, we assume throughout this paper that the sensor and the actuator are merged into a single device, called the *controller*, which fulfills both the roles of estimation and actuation (see Figure 1b). The state of the controller is denoted by C, and assumes values from the set $\mathcal{C}$ of admissible controls. From the viewpoint of information, this simplification amounts to a situation whereby the sensor is connected to the actuator by a noiseless communication channel describing a one-to-one mapping between the input S and the output A of the controller [29].

Using these notations, and the decomposition of Eq.(1), the joint distribution $p(x, x', c)$ describing the causal dependencies between the states of the control graphs can now be constructed. For instance, the complete joint distribution corresponding to the closed-loop graph of Figure 1b is written as

$$p(x, x', c)_{\text{closed}} = p_X(x)p(c|x)p(x'|x, c), \tag{2}$$

whereas the open-loop version of this graph, depicted in Figure 1c, is characterized by a joint distribution of the form

$$p(x, x', c)_{\text{open}} = p_X(x)p_C(c)p(x'|x, c). \tag{3}$$

Following the definition of closed- and open-loop control given before, what distinguishes probabilistically and graphically both control strategies is the presence, for closed-loop control, of a direct correlation link between X and C represented by the conditional probability $p(c|x)$. This correlation can be thought of as a (possibly noisy) communication channel, referred here to as the *sensor* or *measurement* channel, that enables the controller to gather an amount of information identified formally with the *mutual information*

$$I(X; C) = \sum_{x\in\mathcal{X}, c\in\mathcal{C}} p_{XC}(x, c) \log \frac{p_{XC}(x, c)}{p_X(x)p_C(c)}, \tag{4}$$

where $p_{X,C}(x, c) = p_X(x)p(c|x)$. (All logarithms are assumed to the base 2, except where explicitly noted.) Recall that $I(X; C) \geq 0$ with equality if and only if the random variables X and C are statistically independent [30], so that in view of this quantity we are naturally led to define open-loop control with the requirement that $I(X; C) = 0$. Closed-loop control, on the other hand, must be such that $I(X; C) \neq 0$.

As for the actuation part of the control process, the joint distributions of Eqs.(2)-(3) show that it is accounted for by the channel-like probability transition matrix $p(x'|x, c)$. The entries of this *actuation* matrix give the probability that the controlled system in state $X = x$ is actuated to $X' = x'$ given that the controller's state is $C = c$. Henceforth, it will be convenient to think of the control actions indexed by each value of C as a set of *actuation channels*, with memoryless transition matrices

$$p(x'|x)_c = p(x'|x, c), \tag{5}$$

governing the transmission of the random variable X to a target state X'. In terms of the control graphs, such channels are represented similarly as in Figure 1d in order to evidence the fact that the fixed value $C = c$ (filled circle in the graph) enacts a transformation of the random variable X (open circle) to a yet unspecified value associated with the random variable X' (open circle as well). Guided by this graphical representation, we will show in the next section that the overall action of a controller can be decomposed into a series of single conditional actuation actions, or *subdynamics*, triggered by the internal state of C.

Our main concern, in this study, is precisely to characterize the effect of the subdynamics available to a controller on the *entropy* of the initial state X:

$$H(X) = -\sum_{x\in\mathcal{X}} p_X(x) \log p_X(x). \tag{6}$$

In theory, this effect is completely determined by the choice of the initial state X, and the form of the actuation matrices, and can be categorized according to the three following classes of dynamics:

- *One-to-one transitions*. A given control subdynamics specified by $C = c$ conserves the entropy of the initial state X if the corresponding probability matrix $p(x'|x)_c$ is that of a noiseless channel. Permutations or translations of X are examples of this sort of dynamics;
- *Many-to-one transitions*. A control channel $p(x'|x)_c$ may cause a subset of the state space $\mathcal{X}$ to be mapped onto a smaller subset of values for X'. In this case, the corresponding subdynamics is said to be *dissipative* or *volume-contracting* as it decreases entropy of most typical states, for instance, states characterized by non-singular or non-uniform probability distributions;
- *One-to-many transitions*. A channel $p(x'|x)_c$ can also lead $H(X)$ to increase, again in a typical sense, if it is *non-deterministic*, i.e., if it specifies the image of one or more values of X only up to a certain probability different than zero or one. This will be the case, for example, if the actuator is unable to accurately manipulate the dynamics of the controlled system, or if any part of the control system is affected by external and non-controllable systems.

From a strict mathematical point of view, let us note that any non-deterministic channel modeling a source of noise at the level of actuation or estimation can be represented abstractly as a randomly selected deterministic channel with transition matrix containing only zeros and ones. The outcome of a random variable undisclosed to the controller can be thought of as being responsible for the choice of the channel to use. Figure 2 shows specifically how this can be done by supplementing our original control graphs of Figure 1 with an exogenous and non-controllable random variable Z in order to 'purify' the channel considered (actuation or estimation). For the actuation channel, as for instance, the purification condition simply refers to the following properties:

- The mapping from X to X' conditioned on the values c and z, as described by the extended transition matrix $p(x'|x, c, z)$, is deterministic for all $c \in \mathcal{C}$ and $z \in \mathcal{Z}$;

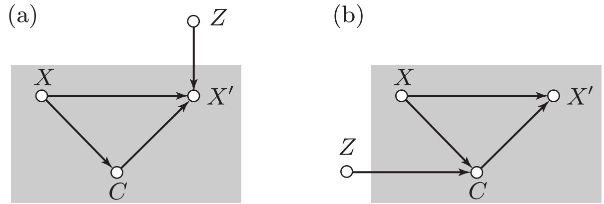

Fig. 2. Control diagrams illustrating the purification procedure for (a) the actuation channel, and (b) the sensor channel. Purifying a channel, for instance the sensor channel, simply means that knowing the value of X and Z enables one to know with probability one the value of C. However, discarding (viz, tracing out) any information concerning Z leaves us with some uncertainty as to which C is reached from a given value for X.

- When traced out of Z, $p(x'|x,c,z)$ reproduces the dynamics of $p(x'|x,c)$, i.e.,

$$p(x'|x,c) = \sum_{z\in\mathcal{Z}} p(x'|x,c,z)p_Z(z), \tag{7}$$

for all x', $x \in \mathcal{X}$, and all $c \in \mathcal{C}$.

III. Conditional analysis

To complement the material introduced in the previous section, we now present a technique for analyzing the control graphs that emphasizes further the conceptual importance of the actuation channel and its graphical representation. The technique is based on a useful symmetry of Figure 1c that enables us to separate the effect of the random variable X in the actuation matrix from the effect of the control variable C. From one perspective, the open-loop decomposition

$$p_{X'}(x')_{\text{open}} = \sum_c p_C(c) \left[\sum_x p(x'|x,c)p_X(x) \right] \tag{8}$$

suggests that an open-loop control process can be decomposed into an ensemble of actuations, each one indexed by a particular value c that takes the initial distribution $p_X(x)$ to a conditional distribution (first sum in parentheses)

$$p(x'|c)_{\text{open}} = \sum_{x\in\mathcal{X}} p(x'|x,c)p_X(x). \tag{9}$$

The final marginal distribution $p_{X'}(x')_{\text{open}}$ is then obtained by evaluating the second sum in Eq.(8), thus averaging $p(x'|c)_{\text{open}}$ over the control variable. From another perspective, Eq.(8), reordered as

$$p_{X'}(x')_{\text{open}} = \sum_x p_X(x) \left[\sum_c p(x'|x,c)p_C(c) \right], \tag{10}$$

indicates that the overall action of a controller can be seen as transmitting X through an 'averaged' channel (sum in parentheses) whose transition matrix is given by

$$p(x'|x) = \sum_{c\in\mathcal{C}} p(x'|x,c)p_C(c). \tag{11}$$

In the former perspective, each actuation subdynamics represented by the control graph of Figure 1d can be characterized by a *conditional open-loop entropy reduction* defined by

$$\Delta H^c_{\text{open}} = H(X) - H(X'|c)_{\text{open}} \tag{12}$$

where

$$H(X'|c) = -\sum_{x'\in\mathcal{X}} p(x'|c)\log p(x'|c). \tag{13}$$

(Subscripts of H indicate from which distribution the entropy is to be calculated.) In the latter perspective, the entropy reduction associated with the unconditional transition from X to X' is simply the *open-loop entropy reduction*

$$\Delta H_{\text{open}} = H(X) - H(X')_{\text{open}} \tag{14}$$

which characterizes the control process as a whole, without regard to any knowledge of the controller's state.

For closed-loop control, the decomposition of the control action into a set of conditional actuations seems *a priori* inapplicable, for the controller's state itself depends on the initial state of the controlled system, and thus cannot be fixed at will. Despite this fact, one can use the Bayesian rule of statistical inference

$$p(x|c) = \frac{p(c|x)p_X(x)}{p_C(c)}, \tag{15}$$

where

$$p_C(c) = \sum_{x\in\mathcal{X}} p(c|x)p_X(x), \tag{16}$$

to invert the dependency between X and C in the sensor channel so as to rewrite the closed-loop decomposition in the following form:

$$p_{X'}(x')_{\text{closed}} = \sum_c p_C(c) \left[\sum_x p(x'|x,c)p(x|c) \right]. \tag{17}$$

By comparing this last equation with Eq.(8), we see that a closed-loop controller is essentially an open-loop controller acting on the basis of $p(x|c)$ instead of $p_X(x)$ [31]. Thus, given that c is fixed, a closed-loop equivalent of Eq.(12) can be calculated simply by substituting $p_X(x)$ with $p(x|c)$, thereby obtaining

$$\Delta H^c_{\text{closed}} = H(X|c) - H(X'|c) \tag{18}$$

for all c.

The rationale for decomposing a closed-loop control action into a set of conditional actuations can be justified by observing naively that *a closed-loop controller, after the estimation step, can be thought of as an ensemble of open-loop controllers acting on a set of estimated states.* In other words, what differentiates open-loop and closed-loop control from the viewpoint of the actuator is the fact that, for the former strategy, a given control action selected by $C = c$ transforms all the values x contained in the *support* of X, i.e., the set

$$\text{supp}(X) = \{x \in \mathcal{X} : p_X(x) > 0\}, \tag{19}$$

whereas for the latter strategy, namely closed-loop control, the same actuation only affects the support of the posterior distribution $p(x|c)$ associated with $X|c$, the random variable X conditioned on the outcome c . This is so because the decision as to which control value is used has been determined according to the observation of specific values of X which are in turn affected by the chosen control value. By combining the influence of all the control values, we thus have that information gathered

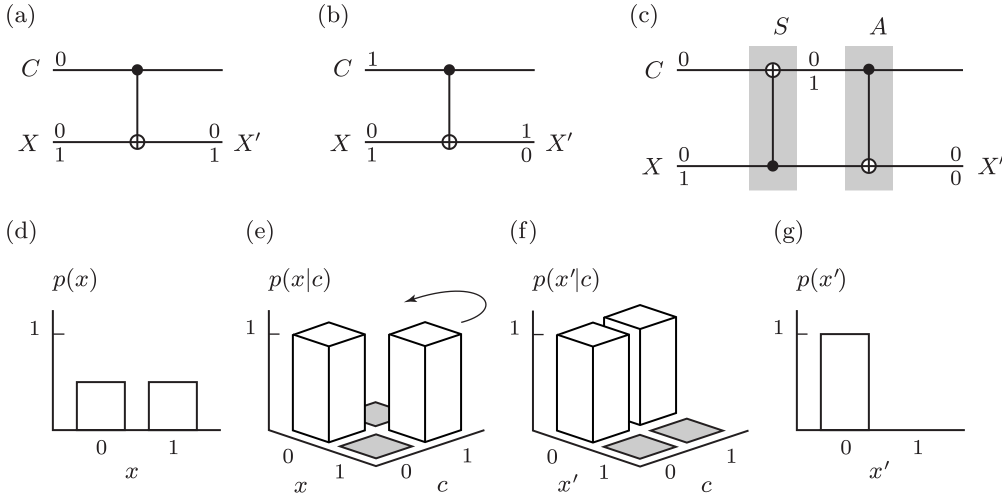

Fig. 3. Controlled-NOT controller. (a) Boolean circuit illustrating the effect of the controller's state $C = 0$ on the input states $0, 1$ of the controlled system X (identity in this case). (b) Control action triggered by $C = 1$ (swapping). (c) Complete control system with sensor S and actuator A. Note that the sensor itself is modeled by a CNOT gate. (d)-(g) State of the controlled system at different stages of the control depicted in the spirit of conditional analysis. (d) A uniformly distributed input state X is measured by the sensor in such a way that the conditional random variable $X|c$ is deterministic (e). (f) The control action triggered by C has the effect of swapping the values x for which $C = 1$. (g) Deterministic probability distribution for the final state X' upon averaging over C.

by the sensor affects the entire control process by inducing a *covering* of the support space

$$\mathrm{supp}(X) = \bigcup_{c\in\mathcal{C}} \mathrm{supp}(X|c), \tag{20}$$

in such a way that values $x \in \mathrm{supp}(X|c_1)$, for a fixed $c_1 \in \mathcal{C}$, are controlled by the corresponding actuation channel $p(x'|x, C = c_1)$, while other values in $\mathrm{supp}(X|c_2)$ are controlled using $p(x'|x, C = c_2)$, and so on for all $c_i \in \mathcal{C}$. This is manifest if one compares Eqs.(8) and (17). Note that a particular value x included in $\mathrm{supp}(X)$ may be actuated by many different control values if it is part of more than one 'conditional' support $\mathrm{supp}(X|c)$. Hence the fact that Eq.(20) only specifies a covering, and not necessarily a partition constructed from non-overlapping sets. Whenever this occurs, we say that the control is *mixing*.

To illustrate the above ideas about subdynamics applied to conditional subsets of $\mathcal{X}$ in a more concrete setting, we proceed in the next paragraph with a basic example involving the control of a binary state system using a controller restricted to use permutations as actuation rules [1]. This example will be used throughout the article as a test situation for other concepts.

Example 1: Let C be a binary state controller acting on a bit X by means of a so-called controlled-NOT (CNOT) logical gate. As shown in the circuits of Figures 3a-b, the state X, under the action of the gate, is left intact or is negated depending on the control value:

$$x' = \begin{cases} x, & \text{if } c = 0 \\ x \oplus 1, & \text{if } c = 1. \end{cases} \tag{21}$$

($\oplus$ stands for modulo 2 addition.) Furthermore, assume that the controller's state is determined by the outcome of a 'perfect' sensor which can be modeled by another CNOT gate such that $C = X$ when C is initially set to 0 (Figure 3c). As a result of these actuation rules, it can be verified that $\Delta H^c_{\text{open}} = \Delta H^c_{\text{closed}} = 0$, and so the application of a single open- or closed-loop control action cannot increase the uncertainty $H(X)$. In fact, whether the subdynamics is applied in an open- or closed-loop fashion is irrelevant here: a permutation is just a permutation in either cases. Now, since $C = X$, we have that the random variable X conditioned on $C = c$ must be equal to c with probability one. For closed-loop control, this implies that the value $X = 0$, which is the only element of $\mathrm{supp}(X|C = 0)$, is kept constant during actuation, whereas the value $X = 1$ in $\mathrm{supp}(X|C = 1)$ is negated to 0 in accordance with the controller's state $C = 1$ (Figure 3e). Under this control action, the conditional random variable $X'|c$ is forced to assume the same deterministic value for all c, implying that X' must be deterministic as well, regardless of the statistics of C (Figures 3f-g). Therefore, $H(X')_{\text{closed}} = 0$. In contrast, the application of the same actuation rules in an open-loop fashion transform the state X to a final state having, at best, no less uncertainty than what is initially specified by the statistics of X, i.e., $H(X')_{\text{open}} \geq H(X)$. □

IV. Entropic formulation of controllability and observability

The first instance of the general control problem that we now proceed to study involves the dual concepts of controllability and observability. In control theory, the importance of these concepts arises from the fact that they characterize mathematically the input-output structure of a system intended to be controlled, and thereby determine whether a given control task is realizable or not [2], [3]. In short, controllability is concerned with the possibilities and limitations of the actuation channel or, in other words, the class of control dynamics that can be effected by a controller. Observability, on the other hand, is concerned with the set of states which are accessible to estimation given that a particular sensor channel is used. In this section, prompted by preliminary results obtained by Lloyd and Slotine [22], we define entropic analogs of the widely held control-theoretic definitions of controllability and observability, and explore the consequences of these new definitions.

A. Controllability

In its simplest expression, we define a system to be *controllable* at $X = x$ if, for any final specified state $X' = x'$, there exists at least one control input for C driving the controlled system from x to x' [2], [3]. In the case of stochastic systems, like those of interest here, we will say that a state x is *approximately controllable* if there exists an actuation subdynamics connecting x to any values of X' with non-vanishing probability. As an extension of these definitions, some authors also define a system to be *completely controllable* whenever the controllability conditions are verified for all initial states $x \in \mathcal{X}$.

In terms of the actuation matrix, controllability for x, or more precisely *perfect* controllability as opposed to approximate controllability, must correspond to the case for which there exists at least one c such that every value x' is reachable from x with probability 1. Let $\mathcal{C}_x$ denote the set of all control values c for which $p(x'|x,c) = 1$ over all $x' \in \mathcal{X}$ and for x fixed. If we restrict the controller's admissible states to values in $\mathcal{C}_x$, i.e., if $\mathrm{supp}(\mathcal{C}) = \mathrm{supp}(\mathcal{C}_x)$, then as a necessary and sufficient condition for perfect controllability we have the following result. (The result was originally put forward in [22] without a complete proof.)

Theorem 1: A system is perfectly controllable at x if and only if $p(x'|x) \neq 0$ for all x' and there exists a non-empty set $\mathcal{C}$ of control values such that

$$H(X'|x,C) = \sum_{c\in\mathcal{C}} H(X'|x,c)p_C(c) = 0, \qquad (22)$$

where

$$H(X'|x,c) = -\sum_{x'\in\mathcal{X}} p(x'|x,c)\log p(x'|x,c). \qquad (23)$$

Proof: If x is controllable, then for each x' there exists at least one control value $c \in \mathcal{C}_x$ such that $p(x'|x,c) = 1$, and thus $H(X'|x,c) = 0$. As this holds true for all $c \in \mathcal{C}_x$, we must also have that the average conditional entropy over C vanishes. Moreover, the condition $p(x'|x,c) = 1$ for all x', and for at least one c such that $p_C(c) \neq 0$, implies

$$p(x'|x) = \sum_{c\in\mathcal{C}} p(x'|x,c)p_C(c) \neq 0, \qquad (24)$$

so that the direct part of the result is proved. To prove the converse, note that if $p(x'|x) \neq 0$ for a given x', then there is at least one value c for which $p(x'|x,c) \neq 0$. In fact, to be more precise, $p(x'|x,c) = 1$ for these particular values x', x and c. Indeed, the condition $H(X'|x,C) = 0$ implies that the random variable X' conditioned on x and c must assume only one value with probability one for all $c \in \mathrm{supp}(C)$. This is verified for any state value x', so that for all $x' \in \mathcal{X}$ there exists a c such that $p(x'|x,c) = 1$. ∎

From a perspective centered on information, $H(X'|x,C)$ has the desirable feature of being interpretable as the residual uncertainty, or uncontrolled variation, left in the output X' when the controller's state C is chosen with respect to the initial value x [22]. If one regards C as an input to a communication channel and X' as the channel output, then the degree to which the final state X' is controlled by manipulating the controller's state can be identified with the conditional mutual information $I(X';C|x)$. This latter quantity can be expressed either using a formula similar to Eq.(4), or by using the expression

$$I(X';C|x) = H(X'|x) - H(X'|x,C), \qquad (25)$$

which is a conditional version of the chain rule

$$I(X;Y) = H(X) - H(X|Y), \qquad (26)$$

valid for any random variables X and Y.

It is interesting to remark that the two above equations allow for another interpretation of $H(X'|x,C)$. The conditional entropy $H(X|Y)$, entering in (26), is often interpreted in communication theory as representing an information loss (the so-called equivocation of Shannon [25]), which results from substracting the maximum noiseless capacity $I(X;X) = H(X)$ of a communication channel with input X and output Y from the actual capacity of that channel as measured by $I(X;Y)$. In our case, we can apply the same reasoning to Eq.(25), and interpret the quantity $H(X'|x,C)$ as a *control loss* which appears as a negative contribution in the expression of $I(X';C|x)$, the number of bits of accuracy to which specifying the control variable specifies the output state of the controlled system. This means that higher is the quantity $H(X'|x,C)$, then higher is the uncertainty or imprecision associated with the outcome of X' upon application of the control action.

In order to characterize the complete controllability of a system, one may also look at the control loss over the entire state space of X, and, in that respect, define

$$L_C = \min_{p_C(c)} H(X'|X,C) \qquad (27)$$

as the *average control loss* over all input states x. In the above equation, the conditional entropy $H(X'|X,C)$ is obtained by averaging $H(X'|x,C)$ over X. Also, for the quantity L_C to be meaningful, we must now assume that the set of admissible controls contains all the subsets $\mathcal{C}_x \subseteq \mathcal{C}$ used to assess the controllability properties of each value x, so that

$$\mathcal{C} = \bigcup_{x\in\mathcal{X}} \mathcal{C}_x. \qquad (28)$$

with $\mathrm{supp}(\mathcal{C}_x) \neq \emptyset$ for all $x \in \mathcal{X}$. In terms of the average control loss we have that a system is perfectly controllable over the support of X if $L_C = 0$ and $p(x'|x) \neq 0$ for all x'. In any other cases, it is approximately controllable for at least one x. The proof of this result follows essentially by noting that, since discrete entropy is positive definite, the condition $H(X'|X,C) = 0$ necessarily implies $H(X'|x,C) = 0$ for all $x \in \mathrm{supp}(X)$.

The next two results relate the average control loss with other quantities of interest. Control graphs containing the purification of the actuation channel, as depicted in Figure 2, are used throughout the rest of this section.

Theorem 2: Under the assumption that X' is a deterministic random variable conditioned on the values x, c, and z (purification assumption), we have $L_C \leq H(Z)$ with equality if, and only if, $H(Z|X',X,C) = 0$.

Proof: Using the general inequality $H(X) \leq H(X,Y)$, and the chain rule for joint entropies, one may write

$$\begin{aligned} H(X'|X,C) &\leq H(X',Z|X,C) \\ &= H(Z|X,C) + H(X'|X,C,Z). \end{aligned} \tag{29}$$

However, $H(X'|X,C,Z) = 0$, since the knowledge of the triplet (x,c,z) is sufficient to infer the value of X' (see the conditions in Section II). Hence,

$$\begin{aligned} H(X'|X,C) &\leq H(Z|X,C) \\ &= H(Z), \end{aligned} \tag{30}$$

where the last equality follows from the fact that Z is chosen independently of X and C as illustrated in the control graph of Figure 2a. Now, from the chain rule

$$H(X',Z|X,C) = H(X'|X,C) + H(Z|X',X,C), \tag{31}$$

it is clear that equality in the first line of expression (29) is achieved if and only if $H(Z|X',X,C) = 0$. ∎

The result of Theorem 2 demonstrates that the uncertainty associated with the control of the state X is upper bounded by the noise level of the actuation channel measured by the entropy of Z. This agrees well with the fact that one goal of controllers is to protect a system against the effects of its environment so as to ensure that it is minimally affected by noise. In the limit where the control loss vanishes, the state X' of the controlled system should show no variability given that we know the initial state and the control action, even in the presence of actuation noise, and should thus be independent of the random variable Z. This is the essence of the next two results which hold for the same conditions as Theorem 2.

Theorem 3: $L_C = I(X';Z|X,C)$.

Proof: From the chain rule of mutual information, we can easily derive

$$I(X';Z|X,C) = H(X'|X,C) - H(X'|X,C,Z). \tag{32}$$

Thus, $I(X';Z|X,C) = H(X'|X,C)$ if we use again the deterministic property of the random variable $X'|x,c,z$ upon purification of $p(x'|x,c)$. ∎

Theorem 4: $L_C = I(X';X,C,Z) - I(X';X,C)$.

Proof: Using the chain rule of mutual information, we write

$$\begin{aligned} I(X';X,C,Z) &= H(X') - H(X'|X,C,Z) \\ &= H(X') - H(X'|X,C,Z) \\ &\quad + H(X'|X,C) - H(X'|X,C) \\ &= I(X';X,C) + I(X';Z|X,C). \end{aligned} \tag{33}$$

For the last equality, we have used Eq.(32). Now, by substituting $L_C = I(X';Z|X,C)$ from the previous theorem, we obtain the desired result. ∎

As a direct corollary of these two results, we have that a system is completely and perfectly controllable if, and only if, $I(X';Z|X,C)$ is equal to zero or equivalently if, and only if,

$$I(X';X,C,Z) = I(X';X,C). \tag{34}$$

Hence, a necessary and sufficient entropic condition for perfect controllability is that the final state of the controlled system, after the actuation step, is statistically independent of the noise variable Z given X and C. In that case, the 'information' $I(X';Z|X,C)$ conveyed in the form of noise from Z to the controlled system is zero. Another 'common sense' interpretation of this result can be given if the quantity $I(X';Z|X,C)$ is instead viewed as representing the 'information' about X' that has been transferred to the non-controllable state Z in the form of 'lost' correlations.

Interestingly, such a perspective on control systems focusing on noise and information protection reminds us that error-correcting codes are designed just like control systems: the information duplicated by a code, when corrupted by noise, is used to detect errors (sensor step) which are then corrected by enacting specific correcting or erasure actions (actuation step) [25], [32], [33]. This somewhat overlooked aspects of error-correcting codes can be strengthened even further if probabilities accounting for undetected and uncorrected errors are modeled by means of communication channels similar to the sensor and actuation channels. In this context, the issue of determining whether or not a prescribed set of erasure actions is sufficient to correct for errors known to occur is determined by the control loss.

B. Observability

The concept of observability is concerned with the issue of inferring the state X of the controlled system based on some knowledge or data of the state provided by a measurement apparatus, taken here to correspond to C. More precisely, a controlled system is termed *perfectly observable* if the sensor's transition matrix $p(c|x)$ maps no two values of X to a single observational output value c, or in other words if for all $c \in \mathcal{C}$ there exists only one value x such that $p(x|c) = 1$. As a consequence, we have the following result [22]. (We omit the proof which readily follows from well-known properties of entropy.)

Theorem 5: A system with state variable X is perfectly observable, with respect to all observed value $c \in \mathrm{supp}(C)$, if and only if

$$H(X|C) = \sum_{c \in \mathcal{C}} H(X|c) p_C(c) = 0. \tag{35}$$

The information-theoretic analog of a perfectly observable system is a *lossless* communication channel $X \to Y$ characterized by $H(X|Y) = 0$ for all input distributions [29]. As a consequence of this association, we interpret the conditional entropy $H(X|C)$ as the information loss, or *sensor loss*, of the sensor channel, henceforth denoted by L_S. This quantity being defined, we now consider the problem of extending the results on controllability into the domain of observability. Specifically, given the similarity between the average control loss L_C and the sensor loss, do we obtain true results for observability by merely substituting L_C by L_S in Theorems 2 and 3?

The answer, rather deceptively, is no for a simple reason: the fact that a communication channel is lossless has nothing to do with the fact that it can be non-deterministic. A convincing example of this is a communication channel that maps the singleton input set $\mathcal{X} = \{0\}$ to multiple instances of the output set

$\mathcal{C}$ with equal probabilities. This is clearly a non-deterministic channel according to our definition, and, yet, since there is only one possible value for X, the conditional entropy $H(X|c)$ must be equal to zero for all $c \in \mathcal{C}$. Hence, contrary to Theorem 2, there can be no result stating that the observation loss L_S is bounded above by the entropy of the random variable responsible for the non-deterministic properties of the sensor channel. However, we are not far from a similar result: by analyzing the meaning of the sensor loss a bit further, one can realize that the generalization of Theorem 2 for observability may in fact be derived using the 'backward' version of the sensor channel. More precisely, $L_S \leq H(Z_B)$ where Z_B is now the random variable associated with the purification of the transition matrix $p(x|c)$. To prove this result, the reader may revise the proof of Theorem 2, and replace the forward purification condition $H(C|X, Z) = 0$ for the sensor channel by its backward analog $H(X|C, Z_B) = 0$.

To close this section, we present next what is left to generalization of the results on controllability. One example aimed at illustrating the interplay between the controllability and observability properties of a system is also given.

Theorem 6: If the state X is perfectly observable, then $I(X; Z|C) = 0$. (The random variable Z stands for the purification variable of the 'forward' sensor channel $p(c|x)$.)

Proof: The proof is rather straightforward. Since $H(X|C) \geq H(X|C, Z)$, the condition $L_S = 0$ implies $H(X|C, Z) = 0$. Thus by the chain rule

$$I(X; Z|C) = H(X|C) - H(X|C, Z), \tag{36}$$

we conclude with $I(X; Z|C) = 0$. ■

Corollary 7: If $L_S = 0$, then $I(X; C, Z) = I(X; C)$.

The interpretations of the two above results follow closely those given for controllability. We will thus not discuss these results furthermore except to mention that, contrary to the case of controllability, $I(X; Z|C) = 0$ is not a sufficient condition for a system to be observable. This follows simply from the fact that $I(X; Z|C) = 0$ implies $H(X|C) = H(X|C, Z)$, and at this point the purification condition $H(C|X, Z) = 0$ for the sensor channel is of no help to obtain $H(X|C) = 0$.

Example 2: Consider again the control system of Figure 3. Given the actuation rules described by the CNOT logical gate, it can be verified easily that for $X = 0$ or 1, $H(X'|x, C) = 0$ and $p(x'|x) \neq 0$ for all x'. Therefore, the controlled system is completely and perfectly controllable. This implies, in particular, that $\Delta H^c_{\text{open}} = \Delta H^c_{\text{closed}} = 0$, and that the final state of the controlled system may be actuated to a single value with probability 1, as noted before. For the latter observation, note that $X' = x'$ with probability 1 so long as the initial state X is known with probability 1 (perfectly observable). In general, if a system is perfectly controllable (actuation property) *and* perfectly observable (sensor property), then it is possible to perfectly control its state to any desired value with vanishing probability of error. In such a case, we can say that the system is *closed-loop controllable*. □

C. *The case of continuous random variables*

The concept of a deterministic continuous random variable is somewhat ill-defined, and, in any case, cannot be associated with the condition $H(X) = 0$ formally. (Consider, e.g., the peaked distribution $p(x) = \delta(x - x_0)$ which is such that $H(X) = -\infty$.) To circumvent this difficulty, controllability and observability for continuous random variables may be extended via a quantization or coarse-graining of the relevant state spaces [30]. For example, a continuous-state system can be defined to be perfectly controllable at x if for every final destination x' there exists at least one control value c which forces the system to reach a small neighborhood of radius $\Delta > 0$ around x' with probability 1. Equivalently, x can be termed perfectly controllable to accuracy Δ if the variable x^Δ obtained by quantizing $\mathcal{X}$ at a scale Δ is perfectly controllable. Similar definitions involving quantized random variables can also be given for observability. The recourse to the quantized description of continuous variables has the virtue that $H(X^\Delta)$ and $H(X^\Delta|C^\Delta)$ are well-defined functions which cannot be infinite. It is also the natural representation used for representing continuous-state models on computers.

V. Stability and Entropy Reduction

The emphasis in the previous section was on proving upper limits for the control and the observation loss, and on finding conditions for which these losses vanish. In this section, we depart from these quantities to focus our attention on other measures which are interesting in view of the stability properties of a controlled system. How can a system be stabilized to a target state or a target subset (attractor) of states? Also, how much information does a controller need to gather in order to achieve successfully a stabilization procedure? To answer these questions, we first propose an entropic criterion of stability, and try to justify its usefulness for problems of control. In a second step, we investigate the quantitative relationship between the closed-loop mutual information $I(X; C)$ and the gain in stability which results from using information in a control process.

A. *Stochastic stability*

Intuitively, a *stable* system is a system which, when activated in the proximity of a desired operating point, stays relatively close to that point indefinitely in time, even in the presence of small perturbations. In the field of control engineering, there exist several formalizations of this intuition, some less stringent than others, whose range of applications depend on theoretical as well as practical considerations. In the next paragraphs, we present a selection of three important criteria of stability which will be discussed thereafter in the light of information theory.

- *Bounded input-bounded output stability* (BIBO) [34] . A system is BIBO stable if any bounded input signals feeding that system, such as control inputs or environment disturbances, cause an always bounded response for the system's observables. The limitations on the signals can be in the form of a bound on the distance between the actual and desired response, a limitation of the signals' variances, a power limitation, etc.

- *Stability in the sense of Lyapunov* [3], [34] . A state x^* is stable if any discrete-time trajectory $\{x_n\}_{n=0}^{\infty}$ initiated by a point x_0, chosen in a small neighborhood of x^*, stays arbitrarily close to that state at all time steps n. Mathematically, this translates into the following. The state x^* is stable if for every $\varepsilon > 0$, one can find $\delta(\varepsilon) > 0$ such that $\|x_0 - x^*\| \leq \delta$ implies $\|x_n - x^*\| \leq \varepsilon$, for all $n > 0$. ($\|\cdot\|$ is an arbitrary norm to be specified.) The ball of radius ε around x^* if often called the Lyapunov stability region. Also, if
$$\lim_{n\to\infty} \|x_n - x^*\| = 0, \tag{37}$$
then x^* is said to be asymptotically stable. This criteria, obviously, can be generalized to continuous-time trajectories.

- *Relative entropy convergence* [35], [36]. The relative entropy or Kullback-Leibler distance between two probability distributions $p(x)$ and $q(x)$, as defined by
$$D(p||q) = \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)}, \tag{38}$$
is a quantity that is always positive, and vanishes only when $p(x) = q(x)$ for all $x \in \mathcal{X}$. For fixed $q(x)$ is it also a convex function of $p(x)$. This means that in the interior of a closed region of the simplex defined by $D(p||q) \leq d$, where $q(x)$ is fixed, there exist only distributions $p(x)$ whose 'distance' to $q(x)$ is smaller than d [30]. Using this property, we can define a distribution-analog of the Lyapunov stability region by requiring that the probability distribution $p_{X_n}(x_n)$ associated with the state X_n approaches a stable target or limiting distribution $p^*(x)$ within a distance d, i.e.,
$$D(p_{X_{n+1}}(x_{n+1})||p^*(x)) \leq D(p_{X_n}(x_n)||p^*(x)), \tag{39}$$
and $D(p_{X_n}(x_n)||p^*(x)) \leq d$ for all $n > 0$. The generalization to continuous-time dynamics is straightforward here again.

In view of the above definitions, it appears logical to reduce the problem of stabilizing a dynamical system to that of decreasing its entropy, or at least immunizing it from sources of entropy like those associated with environment noise, motion instabilities, and incomplete specification of control conditions. This entropic aspect of stabilization is implicit in all of the above criteria insofar as a probabilistic description of systems focusing on sets of responses, rather than on individual response one at a time, is adopted [37]-[39]. For example, a system is BIBO stable if the uncertainty (viz, entropy) associated with the control inputs is not amplified arbitrarily at the output. Similarly, stability in the sense of Lyapunov implies that a set of initial conditions with entropy proportional to $\log\delta$ is constrained to evolve into states which are generally of lower entropy, especially when the system of interest is asymptotically stable. The same argument applies essentially for the relative entropy criterion: in this case, the initial and final entropy of the controlled system, as measured approximately by the logarithm of the support of $p(x_0)$ and $p(x_n)$, respectively, is most likely to be such that $H(X_0) \geq H(X_n)$ for sufficiently large n. For, again, what is usually sought in controlling a system is to confine its possible responses to lie in a set as small as possible, starting from a wide range of initial states.

Many other evidences can be invoked to support the point that stabilizing a system is fundamentally a problem of entropy reduction. The following is only a partial list:

- Least-squares controllers aimed at minimizing the average squared distance between a Gaussian distributed state X and a target state x^* are minimum entropy designs [21];
- Linear unstable systems, with eigenvalues located in the left half part of the complex plane, in addition to unstable nonlinear systems, having positive Lyapunov exponents, are all characterized by positive entropy rates [40];
- From the standpoint of nonequilibrium thermodynamics a system is termed stable if there is more overall entropy dissipation in the system than there is entropy generation [9], [35], [39].

In the light of all these points, we propose to study the two following problems. First, given the initial state X and its entropy $H(X)$, a set of actuation subdynamics, and the type of controller (open- or closed-loop), what is the maximum entropy reduction achievable during the controlled transition from X to X'? Second, what is the quantitative relationship, if there exists one, between the maximal open-loop entropy reduction and the closed-loop entropy reduction?

Note in relation to these questions that, for control purposes, it does not suffice to reduce the entropy of X' conditionally on the state of another system (the controller in particular). For instance, the fact that $H(X'|C)$ vanishes for a given controller acting on a system does not imply by itself that $H(X')$ must vanish as well, or that X' is stabilized. What is required for control is that actuators modify the dynamics of the system intended to be controlled by acting directly on it, so as to reduce the marginal entropy $H(X')$. This unconditional aspect of stability has been discussed in more details in [1].

B. Open-loop control optimality

Using the concavity property of entropy, and the fact that ΔH_{open} is upper bounded by the maximum of ΔH^c_{open} over all control values c, we show in this section that the maximum decrease of entropy achieved by a particular subdynamics of control variable
$$\hat{c} = \arg\max_{c\in\mathcal{C}} \Delta H^c_{\text{open}} \tag{40}$$
is *open-loop optimal* in the sense that no random (i.e., nondeterministic) choice of the controller's state can improve upon that decrease. More precisely, we have the following results. (Theorem 9 was originally stated without a proof in [1].)

Lemma 8: For any initial state X, the open-loop entropy reduction ΔH_{open} satisfies
$$\Delta H_{\text{open}} \leq \Delta H^C_{\text{open}}, \tag{41}$$
where
$$\begin{aligned}\Delta H^C_{\text{open}} &= \sum_{c\in\mathcal{C}} p_C(c) \Delta H^c_{\text{open}} \\ &= H(X) - H(X'|C)_{\text{open}}\end{aligned} \tag{42}$$

with ΔH^c_{open} defined as in Eq.(12). The equality is achieved if and only if $I(X';C)=0$.

Proof: Using the inequality $H(X') \leq H(X'|C)$, we write directly

$$\begin{aligned}\Delta H_{\text{open}} &= H(X) - H(X')_{\text{open}} \\ &\leq H(X) - H(X'|C)_{\text{open}}.\end{aligned} \tag{43}$$

Now, let us prove the equality part. If C is statistically independent of X', then $H(X'|C)=H(X')$, and

$$\Delta H_{\text{open}} = \Delta H^C_{\text{open}}. \tag{44}$$

Conversely, the above equality implies $H(X'|C)=H(X')$, and thus we must have that C is independent of X'. ■

Theorem 9: The entropy reduction achieved by a set of actuation subdynamics used in open-loop control is always such that

$$\Delta H_{\text{open}} \leq \max_{c\in\mathcal{C}} \Delta H^c_{\text{open}}, \tag{45}$$

for all $p_X(x)$. The equality can always be achieved for the deterministic controller $C=\hat{c}$, with $\hat{c}$ defined as in Eq.(40).

Proof: The average conditional entropy $H(X'|C)$ is always such that

$$\min_{c\in\mathcal{C}} H(X'|c) \leq \sum_{c\in\mathcal{C}} p_C(c) H(X'|c). \tag{46}$$

Therefore, making use of the previous lemma, we obtain

$$\begin{aligned}\Delta H_{\text{open}} &\leq \Delta H^C_{\text{open}} \\ &\leq H(X) - \min_{c\in\mathcal{C}} H(X'|c) \\ &= \max_{c\in\mathcal{C}} \Delta H^c_{\text{open}}.\end{aligned} \tag{47}$$

Also, note that if $C=\hat{c}$ with probability 1, then the two above inequalities are saturated since in this case $I(X';C)=0$ and $\Delta H^C_{\text{open}} = \Delta H^{\hat{c}}_{\text{open}}$. ■

An open-loop controller or a control strategy is called *pure* if the control random variable C is deterministic, i.e., if it assumes only one value with probability 1. An open-loop controller that is not pure is called *mixed*. (We also say that a mixed controller activates a mixture of control actions.) In view of these definitions, what we have just proved is that a pure controller with $C=\hat{c}$ is necessarily optimal; any mixture of the control variable either achieves the maximum entropy decrease prescribed by Eq.(45) or yields a smaller value. As shown in the next example, this is so even if the actuation subdynamics used in the control process are deterministic.

Example 3: For the CNOT controller of Example 1, we noted that $H(X')_{\text{open}} = H(X)$, or equivalently that $\Delta H_{\text{open}} = 0$, only *at best*. To be more precise, $\Delta H_{\text{open}} = 0$ only if a pure controller is used or if $H(X)=1$ bit (already at maximum entropy). If the control is mixed, and if $H(X)<1$ bit, then ΔH_{open} must necessarily be negative. This is so because uncertainty as to which actuation rule is used must imply uncertainty as to which state the controlled system is actuated to. □

Note that purity alone is not a sufficient condition for open-loop optimality, nor it is a necessary one in fact. To see this, note on the one hand that a pure controller having

$$C = \arg\min_{c\in\mathcal{C}} \Delta H^c_{\text{open}} \tag{48}$$

with probability one is surely not optimal, unless all entropy reductions ΔH^c_{open} have the same value. On the other hand, to prove that a mixed controller can be optimal, note that if any subset $\mathcal{C}_o \subseteq \mathcal{C}$ of actuation subdynamics is such that $p(x'|c) = p_{X'}(x')$, and ΔH^c_{open} assumes a constant value for all $c \in \mathcal{C}_o$, then one can build an optimal controller by choosing a non-deterministic distribution $p(c)$ with $\text{supp}(C) = \mathcal{C}_o$.

C. Closed-loop control optimality

The distinguishing characteristic of an open-loop controller is that it usually fails to operate efficiently when faced with uncertainty and noise. An open-loop controller acting independently of the state of the controlled system, or solely based on the statistical information provided by the distribution $p_X(x)$, cannot reliably determine which control subdynamics is to be applied in order for the initial (*a priori* unknown) state X to be propagated to a given target state. Furthermore, an open-loop control system cannot compensate actively in time for any disturbances that add to the actuator's driving state (actuation noise). To overcome these difficulties, the controller must be adaptive; that is to say, it must be capable of estimating the unpredictable features of the controlled system during the control process, and must be able to use the information provided by estimation to decide of specific control actions, just as in closed-loop control.

A basic closed-loop controller was presented in Example 1. For this example, we noted that the perfect knowledge of the initial state's value ($X=0$ or 1) enabled the controller to decide which actuation subdynamics (identity or permutation) is to be used in order to actuate the system to $X'=0$ with probability 1. The fact that the sensor gathers $I(X;C)=H(X)$ bits of information during estimation is a necessary condition for this specific controller to achieve $\Delta H_{\text{closed}} = 0$, since having $I(X;C) < H(X)$ may result in generating the value $X'=1$ with non-vanishing probability. In general, just as a subdynamics mapping the input states $\{0,1\}$ to the single value $\{0\}$ would require no information to force X' to assume the value 0, we expect that the closed-loop entropy reduction should not only depend on $I(X;C)$, the effective information available to the controller, but should also depend on the reduction of entropy attainable by open-loop control. The next theorem, which constitutes the main result of this work, embodies exactly this statement by showing that one bit of information gathered by the controller has a maximum value of one bit in the improvement of entropy reduction that closed-loop gives over open-loop control.

Theorem 10: The amount of entropy

$$\Delta H_{\text{closed}} = H(X) - H(X')_{\text{closed}} \tag{49}$$

that can be extracted from a system with given initial state X by using a closed-loop controller with fixed set of actuation subdynamics satisfies

$$\Delta H_{\text{closed}} \leq \Delta H^{\max}_{\text{open}} + I(X;C). \tag{50}$$

where

$$\Delta H_{\text{open}}^{\max} = \max_{p_X(x)\in\mathcal{P}, c\in\mathcal{C}} \Delta H_{\text{open}}^{c} \tag{51}$$

is the maximum entropy decrease that can be obtained by (pure) open-loop control over *any* input distribution chosen in the set $\mathcal{P}$ of all probability distributions.

A proof of the result, based on the conservation of entropy for closed systems, was given in [1] following results found in [41], [42]. Here, we present an alternative proof based on conditional analysis which has the advantage over our previous work to give some indications about the conditions for equality in (50). Some of these conditions are derived in the next section.

Proof: Given that $\Delta H_{\text{open}}^{\max}$ is the optimal entropy reduction for open-loop control over any input distribution, we can write

$$H(X')_{\text{open}} \geq H(X) - \Delta H_{\text{open}}^{\max}. \tag{52}$$

Now, using the fact that a closed-loop controller is formally equivalent to an ensemble of open-loop controllers acting on the conditional supports $\operatorname{supp}(X|c)$ instead of $\operatorname{supp}(X)$, we also have for all $c \in \mathcal{C}$

$$H(X'|c)_{\text{closed}} \geq H(X|c) - \Delta H_{\text{open}}^{\max}, \tag{53}$$

and, on average,

$$H(X'|C)_{\text{closed}} \geq H(X|C) - \Delta H_{\text{open}}^{\max}. \tag{54}$$

That $\Delta H_{\text{open}}^{\max}$ must enter in the lower bounds of $H(X')_{\text{open}}$ *and* $H(X')_{\text{closed}}$ can be explained in other words by saying that each conditional distribution $p(x|c)$ is as a legitimate input distribution for the initial state of the controlled system. It is, in any cases, an element of $\mathcal{P}$. This being said, notice now that $H(X') \geq H(X'|C)$ implies

$$H(X')_{\text{closed}} \geq H(X|C) - \Delta H_{\text{open}}^{\max}. \tag{55}$$

Hence, we obtain

$$\begin{aligned} \Delta H_{\text{closed}} &\leq H(X) - H(X|C) + \Delta H_{\text{closed}}^{\max} \\ &= I(X;C) + \Delta H_{\text{closed}}^{\max}, \end{aligned} \tag{56}$$

which is the desired upper bound. To close the proof, note that $\Delta H_{\text{open}}^{\max}$ cannot be evaluated using the initial distribution $p_X(x)$ alone because the maximum reduction of entropy in open-loop control starting from $p_X(x)$ may differ from the reduction of entropy obtained when some actuation channel is applied in closed-loop to $p(x|c)$. See [43] for a specific example of this. ■

The above theorem enables us to finally understand all the results of Example 1. As noted already, since the actuation subdynamics consist of permutations, we have $\Delta H_{\text{open}}^{\max} = 0$ for any distribution $p_X(x)$. Thus, we should have $\Delta H_{\text{closed}} \leq I(X;C)$. For the particular case studied where $C = X$, the controller is found to be *optimal*, i.e., it achieves the maximum possible entropy reduction $\Delta H_{\text{closed}} = I(X;C)$. This proves, incidentally, that the bound of inequality (50) is tight. In general, we may define a control system to be optimal in terms of information if the gain in stability obtained by substracting $H(X')_{\text{open}}$ from $H(X')_{\text{closed}}$ is exactly equal to the sensor mutual information $I(X;C)$. Equivalently, a closed-loop control system is optimal if its *efficiency* η, defined by

$$\eta = \frac{H(X')_{\text{open}} - H(X')_{\text{closed}}}{I(X;C)}, \tag{57}$$

is equal to 1.

Having determined that optimal controllers do exist, we now turn to the problem of finding general conditions under which a given controller is found to be either optimal ($\eta = 1$) or sub-optimal ($\eta < 1$). By analyzing thoroughly the details of the proof of Theorem 10, one can note that the assessment of the condition $I(X';C) = 0$, which was the necessary and sufficient condition for open-loop optimality, is not sufficient here to conclude that a closed-loop controller is optimal. This comes as a result of the fact that not all control subdynamics applied in a closed-loop fashion are such that $\Delta H_{\text{closed}}^{c} = \Delta H_{\text{open}}^{\max}$ in general. Therefore the average final condition entropy $H(X'|C)_{\text{closed}}$ need not necessarily be equal to the bound imposed by inequality (54). However, in a scenario where the entropy reductions $\Delta H_{\text{open}}^{c}$ and $\Delta H_{\text{closed}}^{c}$ are both equal to a constant for all control subdynamics, then we effectively recover an analog of the open-loop optimality condition, namely that a zero mutual information between the controller and the controlled system *after* actuation is a necessary and sufficient condition for optimality.

Theorem 11: Under the condition that, for all $c \in \mathcal{C}$,

$$\Delta H_{\text{open}}^{c} = \Delta H_{\text{closed}}^{c} = \Delta H, \tag{58}$$

where ΔH is a constant, then a closed-loop controller is optimal if and only if $I(X';C) = 0$.

Proof: To prove the sufficiency part of the theorem, note that the constancy condition (58) implies that the minimum for $H(X')_{\text{open}}$ equals $H(X) - \Delta H$. Similarly, closed-loop control must be such that

$$H(X'|C)_{\text{closed}} = H(X|C) - \Delta H. \tag{59}$$

Combining these results with the fact that $I(X';C) = 0$, or equivalently that

$$H(X')_{\text{closed}} = H(X'|C)_{\text{closed}}, \tag{60}$$

we obtain

$$\begin{aligned} H(X')_{\text{open}}^{\min} - H(X')_{\text{closed}} &= H(X) - H(X|C) \\ &= I(X;C). \end{aligned} \tag{61}$$

To prove the converse, namely that optimality under condition (58) implies $I(X';C) = 0$, notice that Eq.(59) leads to

$$\begin{aligned} H(X')_{\text{open}}^{\min} - H(X'|C)_{\text{closed}} &= H(X) - H(X|C) \\ &= I(X;C). \end{aligned} \tag{62}$$

Hence, given that we have optimality, i.e., given Eq.(61), then X' must effectively be independent of C. ■

Example 4: Consider again the now familiar CNOT controller. Let us assume that instead of the perfect sensor channel $C = X$, we have a binary symmetric channel such that $p(c = x|x) = 1 - e$ and $p(c = x \oplus 1|x) = e$ where $0 \leq e \leq 1$, i.e., an error in the transmission occurs with probability e [30]. The mutual information for this channel is readily calculated to be

$$\begin{aligned} I(X;C) &= H(C) - \sum_{x\in\{0,1\}} p(x)H(C|x) \\ &= H(C) - H(e), \end{aligned} \tag{63}$$

where

$$H(e) = -e\log e - (1-e)\log(1-e) \tag{64}$$

is the binary entropy function. By proceeding similarly as in Example 1, the distribution of the final controlled state can be calculated. The solution is $p_{X'}(0) = 1 - e$ and $p_{X'}(1) = e$, so that $H(X') = H(e)$ and

$$\Delta H_{\text{closed}} = H(X) - H(e). \tag{65}$$

By comparing the value of ΔH_{closed} with the mutual information $I(X;C)$ (recall that $\Delta H_{\text{open}}^{\max} = 0$), we arrive at the conclusion that the controller is optimal for $e = 0$, $e = 1$ (perfect sensor channel), and for $H(X) = 1$ (maximum entropy state). In going through more calculations, it can be shown that these cases of optimality are all such that $I(X';C) = 0$. □

D. Continuous-time limit

In an attempt to derive a differential analog of the closed-loop optimality theorem for systems evolving continuously in time, one may be inclined to proceed as follows: sample the state, say $X(t)$, of a controlled system at two time instants separated by some (infinitesimal) interval Δt, and from there directly apply inequality (50) to the open- and closed-loop entropy reductions associated with the two end-points $X(t)$ and $X(t + \Delta t)$ using $I(X(t); C(t))$ as the information gathered at time t . However sound this approach might appear, it unfortunately proves to be inconsistent for many reasons. First, although one may obtain well-defined rates for $H(X(t))$ in the open- or closed-loop regime, the quantity

$$\lim_{\Delta t\to 0} \frac{I(X(t); C(t))}{\Delta t} \tag{66}$$

does not constitute a rate, for $I(X(t); C(t))$ is not a differential element which vanishes as Δt approaches 0. Second, our very definition of open-loop control, namely the requirement that $I(X;C)$ be equal to 0 prior to actuation, fails to apply for continuous-time dynamics. Indeed, open-loop controllers operating continuously in time must always be such that $I(X(t); C(t)) \neq 0$ if purposeful control is to take place. Finally, are we simply legitimized to extend a result derived in the context of a Markovian or memoryless model of controllers to sampled continuous-time processes, even if the sampled version of such processes has a memoryless structure? Surely, the answer is no.

To overcome these problems, we suggest the following conditional version of the optimality theorem. Let $X(t-\Delta t)$, $X(t)$ and $X(t + \Delta t)$ be three consecutive sampled points of a controlled trajectory $X(t)$. Also, let $C(t - \Delta t)$ and $C(t)$ be the states of the controller during the time interval in which the state of the controlled system is estimated. (The actuation step is assumed to take place between the time instants t and $t + \Delta t$.) Then, by redefining the entropy reductions as conditional entropy reductions following

$$\Delta H^t = H(X(t)|C^{t-\Delta t}) - H(X(t+\Delta t)|C^{t-\Delta t}), \tag{67}$$

where C^t represents the control history up to time t, we must have

$$\Delta H^t_{\text{closed}} \leq \Delta H^t_{\text{open}} + I(X(t); C(t)|C^{t-\Delta t}). \tag{68}$$

Note that by thus conditioning all quantities with $C^{t-\Delta t}$, we extend the applicability of the closed-loop optimality theorem to any class of control processes, be they memoryless or not. Now, since

$$I(X(t-\Delta t); C(t-\Delta t)|C^{t-\Delta t}) = 0 \tag{69}$$

by definition of the mutual information, we also have

$$\begin{aligned} \Delta H^t_{\text{closed}} \leq\; & \Delta H^t_{\text{open}} + I(X(t); C(t)|C^{t-\Delta t}) \\ & -I(X(t-\Delta t); C(t-\Delta t)|C^{t-\Delta t}). \end{aligned} \tag{70}$$

Hence, by dividing both sides of the inequality by Δt, and by taking the limit $\Delta t \to 0$, we obtain the rate equation

$$\dot{H}_{\text{closed}} \leq \dot{H}_{\text{open}} + \dot{I}. \tag{71}$$

This equation relates the rate at which the conditional entropy $H(X(t)|C^{t-\Delta t})$ is dissipated in time with the rate at which the conditional mutual information $I(X(t); C(t)|C^{t-\Delta t})$ is gathered upon estimation. The difference between the above information rate and the previous pseudo-rate reported in Eq.(66) lies in the fact that $I(X(t); C(t)|C^{t-\Delta t})$ represents the differential information gathered during the *latest* estimation stage of the control process. It does not include past correlations induced by the control history $C^{t-\Delta t}$. This sort of conditioning allows, in passing, a perfectly meaningful re-definition of open-loop control in continuous-time, namely $\dot{I} = 0$, since the only correlations between $X(t)$ and $C(t)$ which can be accounted for in the absence of direct estimation are those due to the past control history.

VI. Applications

A. Proportional controllers

There are several controllers, including automatic flight guidance systems, which have the character of applying a control signal with amplitude proportional to the distance or error between some estimate $\hat{X}$ of the state X, and a desired target point x^*. In the control engineering literature, such controllers are designated simply by the term *proportional* controllers [34]. As a simple version of a controller of this type, we study in this section the following system:

$$\begin{aligned} X' &= X - C \\ C &= \hat{X} - x^*, \end{aligned} \tag{72}$$

with all random variables assuming values on the real line. For simplicity, we set $x^* = 0$ and consider two different estimation or sensor channels defined mathematically by

$$C_\Delta = \hat{X} = \left(\left\lfloor \frac{X}{\Delta} \right\rfloor + \frac{1}{2} \right) \Delta, \tag{73}$$

and

$$C_Z = \hat{X} = X + Z, \tag{74}$$

where $Z \sim \mathcal{N}(0, N)$ (Gaussian distribution with zero mean and variance N). The first kind of estimation, Eq.(73), is a coarse-grained measurement of X with a grid of size Δ; it basically allows the controller to 'see' X within a precision Δ, and selects the middle coordinate of each cell of the grid as the control value for C_Δ. The other sensor channel represented by the control state C_Z is simply the Gaussian channel with noise variance N.

Let us start our study of the proportional controller by considering the coarse-grained sensor channel first. If we assume that $X \sim \mathcal{U}(0, \varepsilon)$ (uniform distribution over an interval ε centered around 0), and pose that ε/Δ is an integer, then we must have

$$I(X; C_\Delta) = \log(\varepsilon/\Delta). \tag{75}$$

Now, to obtain $p_{X'}(x')_{\text{closed}}$, note that the conditional random variables $X|c$ defined by conditional analysis are all uniformly distributed over non-overlapping intervals of width ε/Δ, and that, moreover, all of these intervals must be moved under the control law around $X' = 0$ without deformation. Hence, $X' \sim \mathcal{U}(0, \Delta)$, and

$$\begin{aligned} \Delta H_{\text{closed}} &= \log \varepsilon - \log \Delta \\ &= \log(\varepsilon/\Delta). \end{aligned} \tag{76}$$

These results, combined with the fact that $\Delta H_{\text{open}}^{\max} = 0$, prove that the coarse-grained controller is always optimal, at least provided again that ε is a multiple of Δ.

In the case of the Gaussian channel, the situation for optimality is different. Under the application of the estimation law (74), the final state of the controlled system is

$$X' = X - C = X - (X + Z) = -Z, \tag{77}$$

so that $X' \sim Z$. This means that if we start with $X \sim \mathcal{N}(0, P)$, then

$$\begin{aligned} \Delta H_{\text{closed}} &= \frac{1}{2} \log(2\pi e P) - \frac{1}{2} \log(2\pi e N) \\ &= \frac{1}{2} \log \frac{P}{N}, \end{aligned} \tag{78}$$

and

$$I(X; C_Z) = \frac{1}{2} \log \left(1 + \frac{P}{N} \right). \tag{79}$$

Again, $\Delta H_{\text{open}}^{\max} = 0$ (recall that $\Delta H_{\text{open}}^{\max}$ does not depend on the choice of the sensor channel), and so we conclude that optimality is achieved only in the limit where the signal-to-noise ratio goes to infinity. Non-optimality, for this control setup, can be traced back to the presence of some overlap between the different conditional distributions $p(x|c)$ which is responsible for the mixing upon application of the control. As $P/N \to \infty$, the 'area' covered by the overlapping regions decreases, and so is $I(X'; C)$. Based on this observation, we have attempted to change the control law slightly so as to minimize the mixing in the control while keeping the overlap constant and found that complete optimality for the Gaussian channel controller can be achieved if the control law is modified to

$$X' = X - \gamma C, \tag{80}$$

with a *gain* parameter γ set to

$$\gamma = \frac{P}{P + N}. \tag{81}$$

The verification of optimality for this controller is left to the reader.

B. *Noisy control of chaotic maps*

The second application is aimed at illustrating the closed-loop optimality theorem in the context of a controller restricted to use entropy-increasing actuation dynamics, as is often the case in the control of chaotic systems. To this end, we consider the feedback control scheme proposed by Ott, Grebogi and Yorke (OGY) [44] as applied to the *logistic map*

$$x_{n+1} = f(r_n, x_n) = r_n x_n (1 - x_n), \tag{82}$$

where $x_n \in [0, 1]$, and $r_n \in [0, 4]$, $n = 0, 1, 2, \ldots$. In a nutshell, the OGY control method consists in setting the control parameter r_n at each time step n according to

$$\begin{aligned} r_n &= r + \delta r_n \\ \delta r_n &= -\gamma (c_n - x^*) \end{aligned} \tag{83}$$

whenever the estimated state $c_n = \hat{x}_n$ falls into a small control region D in the vicinity of a target point x^*. This target state is usually taken to be an unstable fixed point satisfying the equation $f(r, x^*) = x^*$, where $f(r, x^*)$ is the unperturbed map having $r_n = r$ as a constant control parameter. Moreover, the gain γ is fixed so as to ensure that the trajectory $\{x_n\}_{n=0}^{\infty}$ is stable under the control action. (See [45], [46] for a derivation of the stability conditions for γ based on linear analysis, and [47], [48] for a review of the field of chaotic control.)

Figure 4 illustrates the effect of OGY controller when applied to the logistic map. The plot of Figure 4a shows a typical chaotic trajectory obtained by iterating the dynamical equation (82) with $r_n = r = 3.7825$. Note on this plot the presence of non-recurring oscillations around the unstable fixed point $x^*(r) = (r - 1)/r \simeq 0.7355$. Figure 4b shows the orbit of the same initial point x_0 now stabilized by the OGY controller around x^* for $n \in [50, 150]$. For this latter simulation, and more generally for any initial points in the unit interval, the controller is able to stabilize the state of the logistic map in some region surrounding x^*, provided that γ is a stable gain, and that the sensor channel is not too noisy. To evidence the stability properties of the controller, we have calculated the entropy $H(X_n)$ by constructing a normalized histogram $p_{X_n}(x_n)$ of the positions of a large ensemble of trajectories ($\sim 10^4$) starting at different initial points. The result of this numerical computation is shown

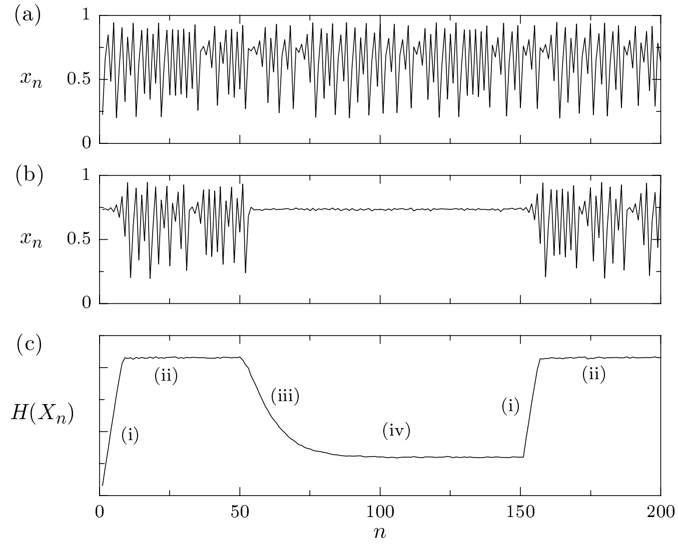

Fig. 4. (a) Typical uncontrolled trajectory of the logistic map with $r = 3.7825$. (b) Controlled trajectory which results from applying the OGY feedback control at time $n = 50$. Note the instant resurgence of instability as the control is switched off at $n = 150$. The gain for this simulation was set to $\gamma = -7.0$, and $D = [0.725, 0.745]$. (c) Entropy $H(X_n)$ (in arbitrary units) associated with the position of the controlled system versus time (see text).

in Figure 4c. On this graph, one can clearly distinguish four different regimes in the evolution of $H(X_n)$, numbered from (i) to (iv), which mark four different regimes of dynamics:

- *Chaotic motion with constant* r (i). Exponential divergence of nearby trajectories initially located in a very small region of the state space. The slope of the linear growth of entropy, the signature of chaos, is probed by the value of the Lyapunov exponent

$$\lambda(r) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln \left| \left. \frac{\partial f(r,x)}{\partial x} \right|_{x_n} \right|; \qquad (84)$$

- *Saturation* (ii). At this point, the distribution of positions $p_{X_n}(x_n)$ for the chaotic system has reached a limiting or equilibrium distribution which nearly fills all the unit interval;
- *Transient stabilization* (iii). When the controller is activated, the set of trajectories used in the calculation of $H(X_n)$ is compressed around x^* exponentially rapidly in time;
- *Controlled regime* (iv). An equilibrium situation is reached whereby $H(X_n)$ stays nearly constant. In this regime, the system has been controlled down to a given residual entropy which specifies the size of the basin of control, i.e., the average distance from x^* to which x_n has been controlled.

It is the size of the basin of control, and, more precisely, its dependence on the amount of information provided by the sensor channel which is of interest to us here. In order to study this dependence, we have simulated the OGY controller, and have compared the value of the residual entropy $H(X_n)$ for two types of sensor channel: the coarse-grained channel $C_n = C_\Delta(X_n)$, and the Gaussian channel $C_n = C_Z(X_n)$.

In the case of the coarse-grained channel, we have found that the distribution of X_n in the controlled regime was well approximated by a uniform distribution of width ε centered around the target point x^*. Thus, the indicator value for the size of the basin of control is taken to correspond to

$$\varepsilon = e^{H(X_n)}, \qquad (85)$$

which, according to the closed-loop optimality theorem, must be such that

$$\varepsilon \geq e^{\lambda^*} \varepsilon_m, \qquad (86)$$

where λ^* is the Lyapunov exponent associated with the r value of the unperturbed logistic map, and where ε_m is the coarse-grained measurement interval or precision of the sensor channel. (All logarithms are in natural base in this section.) To understand the above inequality, note that a uniform distribution for X_n covering an interval of size δ must stretch by a factor $e^{\lambda(r)}$ after one iteration of the map with parameter r. This follows from the fact that $\lambda(r)$ corresponds to an entropy rate of the dynamical system [49]-[54] (see also [37]-[40]), and holds in an average sense inasmuch as the support of X_n is not too small or does not cover the entire unit interval. Now, for open-loop control, it can be seen that if $\lambda(r) > 0$ for all admissible control values r, then no control of the state X_n is possible, and the optimal control strategy must consist in using the smallest Lyapunov exponent $\lambda_{\min}$ available in order to achieve

$$\begin{aligned} \Delta H_{\text{open}}^{\max} &= H(X_n) - H(X_{n+1})_{\text{open}} \\ &= \ln \delta - \ln e^{\lambda_{\min}} \delta \\ &= -\lambda_{\min} < 0. \end{aligned} \qquad (87)$$

In the course of the simulations, we noticed that only a very narrow range of r values were actually used in the controlled regime, which means that $\Delta H_{\text{open}}^{\max}$ can be taken for all purposes to be equal to $-\lambda^*$. At this point, then, we need only to use expression (75) for the mutual information of the coarse-grained channel, substituting Δ with ε_m, to obtain

$$\Delta H_{\text{closed}} \leq -\lambda^* + \ln(\varepsilon/\varepsilon_m). \qquad (88)$$

This expression yields the aforementioned inequality by posing $\Delta H_{\text{closed}} = 0$ (controlled regime).

The plots of Figure 5 present our numerical calculations of ε as a function of ε_m. Each of these plots has been obtained by calculating Eq.(85) using the entropy of the normalized histogram of the positions of about 10^4 different controlled trajectories. Other details about the simulations may be found in the caption. What differentiates the four plots is the fixed point to which the ensemble of trajectories have been stabilized, and, accordingly, the value of the Lyapunov exponent λ^* associated to $x^*(r)$. These are listed in Table 1 and illustrated in Figure 6. One can verify on the plots of Figure 5 that the points of ε versus ε_m all lie above the critical line (solid line in the graphs) which corresponds to the optimality prediction of inequality (86). Also, the relatively small departure of the numerical data from the optimal prediction shows that the OGY controller with the coarse-grained channel is nearly optimal with

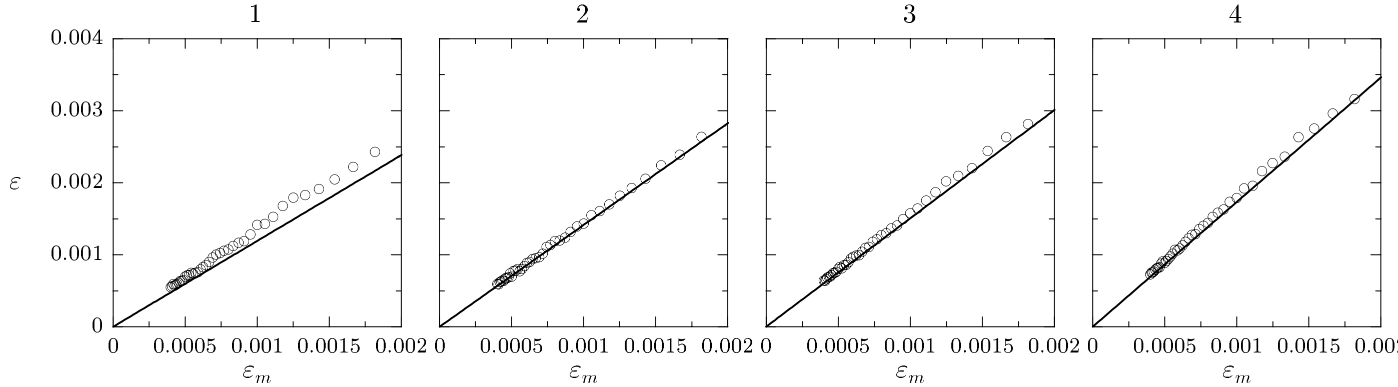

Fig. 5. (Data points) Control interval ε as a function of the effective coarse-grained interval of measurement ε_m for four different target points. (Solid line) Optimal linear relationship predicted by the closed-loop optimality theorem. The values of r and the Lyapunov exponents λ^* associated with the target points are listed in Table 1 and displayed in Figure 6.

TABLE I. Characteristics of the four target points.

Target point	x^*	r	λ^* (base e)
1	0.7218	3.5950	0.1745
2	0.7284	3.6825	0.3461
3	0.7356	3.7825	0.4088
4	0.7455	3.9290	0.5488

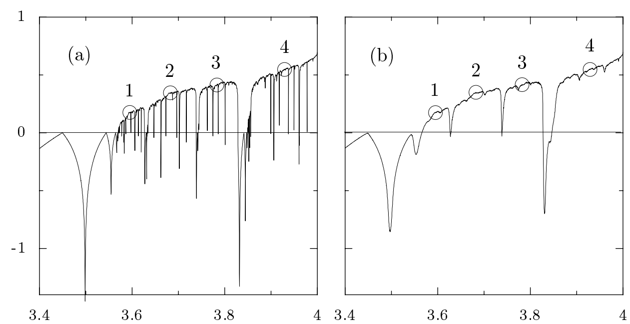

Fig. 6. (a) Lyapunov spectrum $(r, \lambda(r))$ of the logistic map. The positive Lyapunov exponents associated with the four target points listed in Table 1 are located by the circles. The set of r values used during the control spans approximately the diameter of the circles. Note that the few negative values of $\lambda(r)$ close to the λ^*'s are effectively suppressed by the noise in the sensor channel. This is evidenced by the graph of (b) which was obtained by computing the sum (84) up to $N = 2 \times 10^4$ with an additive noise component of very small amplitude. See [55], [56] for more details on this point.

respect to the entropy criterion. This may be explained by noticing that this sort of controller complies with all the requirements of the first class of linear proportional controllers studied previously. Hence, we expect it to be optimal for all precision ε_m, although the fact must be considered that $\Delta H_{\text{open}}^{\max} = -\lambda^*$ is only an approximation. In reality, not all points are controlled with the same parameter r for a given value of ε_m, as shown in Figure 6. Moreover, how ε is calculated explicitly relies on the assumption that the distribution for X_n is uniform. This assumption has been verified numerically; yet, it must also be regarded as an approximation. Taken together, these two approximations may explain the observed deviations of ε from its optimal value.

For what concerns the Gaussian channel, the situation of optimality is also very related to our results about proportional controllers. The results of our simulations, for this type of channel, indicated that the normalized histogram of the controlled positions for X_n is very close to a normal distribution with mean x^* and variance P. As a consequence, we now consider the variance P, which for Gaussian random variables is given by

$$P = \frac{e^{2H(X_n)}}{2\pi e}, \tag{89}$$

as the correlate of the size of the basin of control. For this quantity, the closed-loop optimality theorem with $\Delta H_{\text{closed}} = 0$ yields

$$P \geq (e^{2\lambda^*} - 1)N, \tag{90}$$

where N is the variance of the zero-mean Gaussian noise perturbing the sensor channel.

In Figure 7, we have displayed our numerical data for P as a function of the noise power N. The solid line gives the optimal relationship which results from taking equality in the above expression, and from substituting the Lyapunov exponent associated with one of the four stabilized points listed in Table 1. From the plots of this figure, we verify again that P is lower bounded by the optimal value predicted analytically. However, now it can be seen that P deviates significantly from its optimal value, making clear that the OGY controller driven by the Gaussian noisy sensor channel is not optimal (except in the trivial limit where $N \to 0$). This is in agreement with our proof that linear proportional controllers with Gaussian sensor channel are not optimal in general. On the plots of Fig. 7, it is quite remarkable to see that the data points all converge to straight lines. This suggests that the mixing induced by the controller, the source of non-optimality, can be accounted for simply by modifying our inequality for P so as to obtain

$$P = (e^{2\lambda'} - 1)N. \tag{91}$$

The new exponent λ' can be interpreted as an *effective* Lyapunov exponent; its value is necessarily greater than λ^*, since the chaoticity properties of the controlled system are enhanced by the mixing effect of the controller.

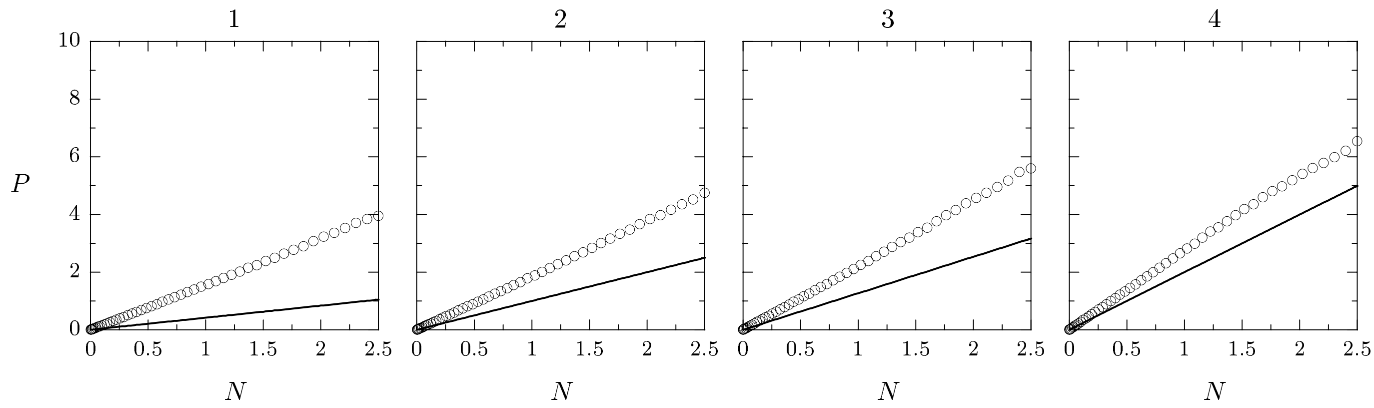

Fig. 7. (Data points) Dispersion P characterizing the basin of attraction of the controlled system as a function of the noise power N introduced in the Gaussian sensor channel. The horizontal and vertical axes are to be rescaled by a factor 10^{-5}. (Solid line) Optimal lower bound.

VII. Concluding remarks

A. Control and thermodynamics

The reader familiar with thermodynamics may have noted some similarity between the functioning of a controllers, when viewed as a device aimed at reducing the entropy of a system, and the so-called Maxwell demon [26]. Such a similarity is not fortuitous: as he wondered about the validity of the second law of thermodynamics, the physicist James Clerk Maxwell imagined the first example of a system whose task, in effect, is to reduce the entropy of another system by putting to use information, and, for this reason, has been the original impetus for this work. In the case of Maxwell's demon, the system to be controlled or 'cooled' is a volume of gas; the entropy to be reduced is the equilibrium thermodynamic entropy of the gas; and the pieces of information gathered by the controller (the demon) are the velocities of the atoms or molecules constituting the gas.

When applied to this scheme, our result on closed-loop optimality can be translated into an absolute limit to the ability of the demon, or any control devices, to convert heat to work. Indeed, consider a feedback controller operating in a cyclic fashion on a system in contact with a heat reservoir at temperature T. According to Clausius law of thermodynamic [57], the amount of heat ΔQ_{closed} extracted by the controller upon reducing the entropy of the controlled system by a concomitant amount ΔH_{closed} must be such that

$$\Delta Q_{\text{closed}} = (k_B T \ln 2)\Delta H_{\text{closed}}. \tag{92}$$

In the above equation, k_B is the Boltzmann constant which provides the necessary conversion between units of energy (Joule) and units of temperature (Kelvin); the constant $\ln 2$ arises because entropy, in physics, is customary defined in base e. From the closed-loop optimality theorem, we then write

$$\begin{aligned}\Delta Q_{\text{closed}} &\leq (k_B T \ln 2)[\Delta H_{\text{open}}^{\max} + I(X;C)]\\ &= \Delta Q_{\text{open}}^{\max} + (k_B T \ln 2) I(X;C),\end{aligned} \tag{93}$$

where $\Delta Q_{\text{open}}^{\max} = (k_B T \ln 2)\Delta H_{\text{open}}^{\max}$. This limit should be compared with analogous results found by other authors on the subject of thermodynamic demons. Consult, as for example, some of the articles reprinted in [26], and especially Szilard's analysis of Maxwell's demon. This classic paper, originally published in [58], contains many premonitory insights about the use of information in control.

It should be remarked that the connection between the problem of Maxwell's demon, thermodynamics, and control is effective only to the extent that Clausius law provides a link between entropy and the physically measurable quantity that is energy. But, of course, the notion of entropy is a more general notion than what is implied by Clausius law; it can be defined in relation to several situations which have no direct relationship whatsoever with physics (e.g., coding or decision theory). This versatility of entropy is implicit here. Our results do not rely on thermodynamic principles, or even physical principles for that matter, to be true. They constitute valid results derived in the context of a general model of control processes whose precise nature is yet to be specified.

B. Entropy and optimal control theory

Consideration of entropy as a measure of dispersion and uncertainty led us to choose this quantity as a control function of interest, but other information-theoretic quantities may well have been chosen instead if different control applications require so. From the point of view of optimal control theory, all that is required is to minimize a desired performance criterion (a cost or a Lyapunov function), such as the distance to a target point or the energy consumption, while achieving some desired dynamic performance (stability) using a set of permissible controls [34], [17]. For example, one may be interested to maximize ΔH_{closed} instead of minimizing this quantity if destabilization (anti-control) or mixing is an issue [59]. As other examples, let us mention the minimization of the relative entropy distance between the distribution of the state of a controlled system and some target distribution [36], the problem of coding [60], as well as the minimization of rate-like functions in decision or game theory [61]-[63], [30].

C. Control and rate distortion theory

The conceptual closeness of optimal control theory and the theory of rate distortion [25], [30] can serve as another basis for an information-theoretic formulation of control. This possibility has not been considered explicitly here, but should surely be

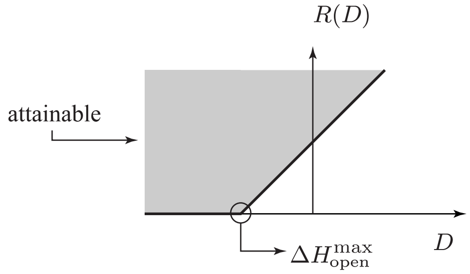

Fig. 8. Rate distortion function $R(D)$ for the general control systems studied in this paper.

investigated in more details. It consists, specifically, in considering a fidelity criterion, say a real-valued function $d(X, X')$ of the initial and final states X and X', and to seek for the least amount of information $R(D)$ needed for a controller to achieve some upper bound D on $d(X, X')$ using a fixed set of actuation dynamics. The function $R(D)$ so defined is known as the *rate distortion* function. A *distortion rate* function may also be defined if what is sought is the maximum $D(R)$ of the performance function which can be attained under a communication constraint $I(X;C) \leq R$. (See [21], [64] for an analogous approach to sensor filters and linear controllers.)

Evidently, different definitions of these quantities may be given along these lines if the performance criterion is to be maximized instead of being minimized. From a formal point of view, minimizing a criterion functional simply amounts to maximizing the same functional with a minus sign. Thus, in this case, the functions $R(D)$ and $D(R)$ should properly be re-defined as follows:

$$\begin{aligned} R(D) &= \min_{p(c|x):d(X,X')\geq D} I(X;C) \\ D(R) &= \max_{p(c|x):I(X;C)\leq R} d(X,X'). \end{aligned} \tag{94}$$

What we have shown in Section V is that, if the performance criterion for control is taken to correspond to the closed-loop entropy reduction, i.e., $d(X, X') = H(X) - H(X')_{\text{closed}}$, then

$$R(D) = \max(0, \Delta H_{\text{open}}^{\max} - D) \tag{95}$$

and

$$D(R) = \Delta H_{\text{open}}^{\max} - R. \tag{96}$$

These two equations are illustrated in Figure 8. Note that $\Delta H_{\text{open}}^{\max}$ is a constant of the problem, since what is varied above is the sensor channel. Similar relations hold if $R(D)$ and $D(R)$ are defined by requiring that the sensor channel is fixed and that an optimal design for the controller is to be found by selecting an appropriate actuation channel.

D. Beyond Markovian models

Many questions pertaining to issues of information and control remain at present unanswered. We have considered in this paper only but the first level of investigation of a much broader and definitive program of research aimed at providing information-theoretic tools for the study of general control systems, such as those involving many interacting components, as well as controllers exploiting non-Markovian features of dynamics (e.g., memory, learning, and adaptation). In a sense, what we have studied can be compared with the memoryless channel of information theory; what is needed in the future is something like a control analog of network information theory. Work is ongoing along this direction.

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